

On the shortfall risk control - a refinement of the quantile hedging method

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December 11, 2015

Abstract

The issue of constructing a risk minimizing hedge under an additional almost-surely type constraint on the shortfall profile is examined. Several classical risk minimizing problems are adapted to the new setting and solved. In particular, the bankruptcy threat of optimal strategies appearing in the classical risk minimizing setting is ruled out. The existence and concrete forms of optimal strategies in a general semimartingale market model with the use of conditional statistical tests are proven. The quantile hedging method applied in [9] and [10] as well as the classical Neyman-Pearson lemma are generalized. Optimal hedging strategies with shortfall constraints in the Black-Scholes and exponential Poisson model are explicitly determined.

Key words: quantile hedging, Neyman-Pearson lemma, shortfall constraints, bankruptcy prohibition, conditional tests.

AMS Subject Classification: 91B30, 91B24, 91B70,

JEL Classification Numbers: G13, G10.

1 Introduction

Let us briefly sketch the classical hedging problem in a stochastic model of financial market. The goal of an investor having an initial capital $x \geq 0$ is to hedge dynamically a given random variable H which represents the payoff of a financial contract at some future date $T > 0$. He is looking for a trading strategy π such that the related portfolio wealth $X_T^{x,\pi}$ at T exceeds H almost surely, i.e.

$$P(X_T^{x,\pi} \geq H) = 1. \quad (1.1)$$

A strategy π satisfying (1.1) is called a hedging or superhedging strategy for H and it is well known that it exists if x is greater than the superhedging price of H . In the opposite case each trading strategy is able to hedge the claim at most partially, i.e. $P(X_T^{x,\pi} \geq H) < 1$, and hence generates the shortfall

$$(H - X_T^{x,\pi})^+ := \max\{0, H - X_T^{x,\pi}\}$$

which is strictly positive with positive probability. The related shortfall risk which appears in that case should be minimized to protect the investor against the resulting loss. There is an extensive literature on minimizing shortfall risk, see for instance [1], [2], [4], [9], [10], [11], [12], [13], [14], [15], [16], [19], with various risk measures accepted by the investor. Four of those measures listed below play a central role in our study. In the quantile hedging approach, introduced in [9], the objective was to maximize the probability of a successful hedge, i.e.

$$\max_{\pi} P(X_T^{x,\pi} \geq H). \quad (1.2)$$

The drawback of (1.2) of neglecting the portfolio performance on the set $\{X_T^{x,\pi} < H\}$ has been partially eliminated in [9] in the generalized version of (1.2) which is

$$\max_{\pi} E[\varphi_{x,\pi}], \quad \text{where} \quad \varphi_{x,\pi} := 1 \wedge \frac{X_T^{x,\pi}}{H}. \quad (1.3)$$

Another optimality criterion was to minimize the weighted expected shortfall, i.e.

$$\min_{\pi} E[l((H - X_T^{x,\pi})^+)], \quad (1.4)$$

where $l : \mathbb{R} \rightarrow \mathbb{R}$ is a so called loss function. The case $l(z) = z$ has been studied in [4] and the general case in [10] and [15]. If $l(z) = az + b, a \geq 0, b \in \mathbb{R}$ then (1.4) can be written as

$$\min_{\pi} \rho(-(H - X_T^{x,\pi})^+), \quad (1.5)$$

where ρ is defined by $\rho(Y) := E[l(-Y)], Y \in L^1$. In this case ρ is a coherent risk measure of a special form, see Section 2 for a precise definition. The general form of (1.5) where $\rho : L^p \rightarrow \mathbb{R}, p > 1$ is a coherent risk measure on L^p has been studied in [14].

The motivation for the present paper arises from the fact that the profile of the shortfall in all the problems (1.2), (1.3), (1.4) and (1.5) remains beyond the trader's control. The preferences of the trader towards the size of the shortfall are not described sufficiently well by the risk measures mentioned above and consequently even a risk minimizing strategy may generate the loss which exceeds the solvency of the trader. So, even the best performance may lead to bankruptcy in finite time! This problem is apparent in the quantile hedging approach because, as shown in [9], the optimal strategy $\tilde{\pi}$ for (1.2) is such that

$$X_T^{x,\tilde{\pi}} = H \mathbf{1}_A,$$

where A is some subset of Ω which depends on x . It follows that the shortfall equals $H \mathbf{1}_{A^c}$ which means that the shortfall risk is completely unhedged on A^c . This depicts the quantile hedging method as a risky tool for minimizing the risk. Although the risk measures in (1.3) and (1.4) are more involved, the problem of an uncontrolled shortfall profile appears there as well. To illustrate that let us consider a call option $H = (S_T - K)^+$ on the underlying asset S in the classical Black-Scholes model with drift α and volatility $\sigma > 0$. It was shown in [9, p. 261] that in the case when $\alpha < \sigma^2$ the optimal strategy $\tilde{\pi}$ for (1.3) generates the wealth

$$X_T^{x,\tilde{\pi}} = (S_T - K)^+ - (S_T - k)^+ - (k - K) \mathbf{1}_{\{S_T > k\}},$$

where $k > K$ is a certain constant which depends on x . Thus the related shortfall equals

$$H - X_T^{x,\tilde{\pi}} = (S_T - k)^+ + (k - K)\mathbf{1}_{\{S_T > k\}}.$$

In particular, it is clear that the shortfall is unbounded on the set $\{S_T > k\}$ which implies a positive ruin probability for each investor regardless of the level of his solvency. An analogous example can be constructed for (1.4) with $l(z) = z$ and the claim $H := \frac{1}{S_T}$.

In this paper we show how to incorporate a relevant shortfall profile into the problems (1.2), (1.3), (1.4) and (1.5) which in turn allows to obviate the drawbacks of optimal strategies mentioned above. The idea is to introduce a shortfall constraint L which is a random variable acting as upper bound for the shortfall and study the problems (1.2), (1.3), (1.4), (1.5) subject to the additional condition

$$P((H - X_T^{x,\pi})^+ \leq L) = 1. \quad (1.6)$$

Since L is of a fairly general form, this setting provides a flexible tool for managing hedging risk and allows to accommodate fully the risk preferences of the investor. In particular, an appropriate choice of a bounded shortfall constraint protects him against a bankruptcy threat. Coming back to the example with a call option mentioned above, let us assume that the trader wants to keep the shortfall below a constant margin $c > 0$. Our general results applied to this particular situation yield explicit solutions to the problems (1.2), (1.3) and (1.4). It turns out that the portfolio wealth of an optimal strategy is a digital combination of two options: $(S_T - K)^+$ and $(S_T - (K + c))^+$. A precise form of the combination depends on the risk measure and the initial capital x . The optimal strategy for (1.2) satisfies

$$X_T^{x,\tilde{\pi}} = (S_T - K)^+ \mathbf{1}_{\{S_T \leq k_1\} \cup \{S_T \geq k_2\}} + (S_T - (K + c))^+ \mathbf{1}_{\{k_1 < S_T < k_2\}},$$

with $K < k_1 \leq K + c$ and $k_2 \geq K + c$. For (1.3) the optimal portfolio is such that

$$X_T^{x,\tilde{\pi}} = (S_T - K)^+ \mathbf{1}_{\{S_T \leq k_3\}} + (S_T - (K + c))^+ \mathbf{1}_{\{S_T > k_3\}},$$

with $k_3 > K$ while for (1.4) with the loss function $l(z) = z$ we obtain

$$X_T^{x,\tilde{\pi}} = (S_T - K)^+ \mathbf{1}_{\{S_T > k_4\}} + (S_T - (K + c))^+ \mathbf{1}_{\{S_T \leq k_4\}},$$

with $k_4 > K$. All the constants above depend on x .

In this paper we characterize optimal solutions for the problems (1.2), (1.3), (1.4), (1.5) under (1.6) for general forms of H and L . For the sake of generality, (1.5) will be analysed for a convex risk measure instead of a coherent one, see Section 2 for definition. Our assumptions concerning the market are rather weak because we require only that the price process (S_t) is a locally bounded semimartingale and that the market is free of arbitrage in the sense that there is no free lunch with vanishing risk NFLVR. This setting enables us to use the dual characterization of the superhedging price proved in [6]. Our investigation relies on a certain restriction of the success ratio $\varphi_{x,\pi}$, defined in (1.3), which is implied by (1.6). It allows to characterize the solutions of (1.2), (1.3), (1.4), (1.5) in terms of certain statistical tests, i.e. $[0, 1]$ -valued random

variables, which exceed a prespecified test φ^* . We call each test of this a conditional test with a rejection threshold φ^* . In the presented framework φ^* is determined by H and L . Our approach is a modification of the celebrated quantile hedging method applied in [9] and [10] and generalizes the results where the shortfall profile was unconstrained. This particular situation corresponds to the condition $L = H$ which generates the trivial rejection threshold $\varphi^* \equiv 0$. In Lemma 4.1 we prove a generalized version of the Neyman-Pearson lemma for conditional statistical tests. This enables us to find the explicit form of optimal solutions in the case when the market is complete and the laws of ZH and ZL are free of positive atoms, where Z stands for the density of the martingale measure, see Proposition 4.3 for details. As a consequence we obtain a precise characterization of optimal strategies in the Black-Scholes and exponential Poisson model.

The paper is organized as follows. In Section 2 we describe the market model and formulate the optimization problems in a precise way. The main results characterizing optimal strategies with shortfall constraints are proven in Section 3. The concept of a conditional statistical test is discussed in Section 4 where also a generalized version of the Neyman-Pearson lemma is proven and its application to determining optimal payoffs is shown. Examples are presented in Section 5.

2 Formulations of the problems

We will consider a continuous time financial market of a general form, studied in [7], where the stock price is given by an \mathbb{R}^d -valued locally bounded semimartingale $(S_t)_{t \in [0, T]}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in [0, T]}, \mathcal{F} = \mathcal{F}_T, P)$ with $T < +\infty$. Trading positions are represented by pairs (x, π) , where $x \geq 0$ stands for an initial capital of the investor and $(\pi_t)_{t \in [0, T]}$ is a predictable, S -integrable stochastic process describing the trading strategy, that is, the wealth allocation between stocks. The resulting *gain process*

$$(\pi \cdot S)_t := \int_0^t \pi(s) dS(s), \quad t \in [0, T],$$

is assumed to be uniformly bounded from below, i.e. there exists a constant $m \geq 0$ such that $(\pi \cdot S)_t \geq -m$ for each $t \in [0, T]$ a.s.. The class of all such strategies will be denoted by Π . If the *wealth process* corresponding to (x, π) , $x \geq 0$, $\pi \in \Pi$ given by

$$X_t^{x, \pi} := x + (\pi \cdot S)_t, \quad t \in [0, T],$$

satisfies $X_T^{x, \pi} \geq 0$, a.s., then (x, π) will be called *admissible*. For the sake of simplicity we assume that the risk-free interest rate equals zero, i.e. $r = 0$, so the value of 1 Euro on a savings account is constant in time. Let us define

$$K := \{X = X_T^{0, \pi} = (\pi \cdot S)_T, \pi \in \Pi\},$$

a set of final portfolio values attained from zero initial capital and a set

$$C := \{Y \in L^\infty(\Omega, \mathcal{F}, P) : Y \leq X \text{ for some } X \in K\}$$

of all bounded random variables dominated by some element of K . The model admits *no free lunch with vanishing risk* NFLVR if

$$\bar{C} \cap L_+^\infty(\Omega, \mathcal{F}, P) = \{0\},$$

where \bar{C} stands for the closure of C in $L^\infty(\Omega, \mathcal{F}, P)$ and $L_+^\infty(\Omega, \mathcal{F}, P)$ consists of all non-negative elements of $L^\infty(\Omega, \mathcal{F}, P)$. The NFLVR condition precludes arbitrage opportunities from the market, which means that there are no risk-free investments generating profits. If NFLVR is violated then there is a sequence of strategies $\pi^n \in \Pi$ such that the corresponding gains satisfy

$$(\pi^n \cdot S)_T \longrightarrow Y, \quad (\pi^n \cdot S)_T \geq Y - \frac{1}{n},$$

for some $0 \neq Y \in \bar{C} \cap L_+^\infty$. Hence $\{\pi^n\}$ realizes a positive profit at T with zero initial cost and asymptotically vanishing risk because $(\pi^n \cdot S)_T \geq -\frac{1}{n}, n = 1, 2, \dots$. Theorem 1 in [6] provides a characterization of NFLVR which is

$$NFLVR \iff \mathcal{Q} \neq \emptyset, \quad (2.1)$$

where \mathcal{Q} stands for the set of all equivalent to P probability measures under which (S_t) is a local martingale. In this paper we will work under the NFLVR condition, so we make the standing assumption $\mathcal{Q} \neq \emptyset$. A dual representation of superhedging prices arising from (2.1), which we describe below, plays a central role in our analysis.

Let $H \geq 0$ be an \mathcal{F}_T -measurable random variable representing a contingent claim with payoff at time T . An admissible strategy $(x, \pi), x \geq 0, \pi \in \Pi$ is a *superhedge* of H if

$$P(X_T^{x, \pi} \geq H) = 1. \quad (2.2)$$

The *superhedging price* of H is defined by

$$p(H) := \inf \left\{ x \geq 0 : \text{there exists } \pi \in \Pi \text{ such that } P(X_T^{x, \pi} \geq H) = 1 \right\}.$$

The dual characterization of $p(H)$ has been proven in [6], see Theorem 9 and Corollary 10.

Theorem 2.1 *Under NFLVR the superhedging price admits the following dual representation*

$$p(H) = \sup_{Q \in \mathcal{Q}} E^Q[H], \quad (2.3)$$

where $E^Q[\cdot]$ stands for the expectation under Q . Moreover, if $x = \sup_{Q \in \mathcal{Q}} E^Q[H] < +\infty$ then there exists $\pi \in \Pi$ such that (x, π) satisfy (2.2).

In the particular case when \mathcal{Q} is a singleton the price of H is given by the expectation of the claim under the unique martingale measure, i.e. $p(H) = E^Q[H]$. If the latter is finite then it follows from Theorem 16 in [6] that the inequality in (2.2) holds as equality. Then the market is *complete* and a hedging strategy satisfying $X_T^{x, \pi} = H$ is called a *replicating strategy*. The concept of superhedging was introduced in [8] where (2.3) was proven in the context of a concrete model driven by a multidimensional diffusion process. The equivalence (2.1) and formula (2.3) can be

generalized to the case when (S_t) is a semimartingale which is not necessarily locally bounded, see Theorem 1.1 and Theorem 5.12 in [7]. In this case the set \mathcal{Q} , however, must be replaced with the set of all sigma-martingale measures and the superhedging strategy against H for $x = \sup_{Q \in \mathcal{Q}} E^Q[H] < +\infty$ exists in an extended class of strategies satisfying certain technical conditions, see Theorem 5.5 in [7]. The choice of a model with a locally bounded semimartingale (S_t) seems to keep a good balance between the generality of our consideration and clarity of presentation.

Let us consider the situation when the initial capital x does not allow to superhedge H , i.e. satisfies $0 < x < p(H)$. It follows from Theorem 2.1 that then (2.2) is violated and hence each strategy $\pi \in \Pi$ is biased by hedging risk with positive probability, that is

$$P((H - X_T^{x,\pi})^+ > 0) > 0.$$

The problem of the investor is to minimize the risk which is quantified by a properly chosen risk measure. A component of the risk measures considered in the sequel is a condition describing a maximal size of the shortfall $(H - X_T^{x,\pi})^+$ which is acceptable by the trader. A *shortfall constraint* is defined by an \mathcal{F}_T -measurable random variable L satisfying the condition

$$0 \leq L \leq H, \tag{2.4}$$

and it constitutes those admissible strategies as acceptable which satisfy

$$P((H - X_T^{x,\pi})^+ \leq L) = 1. \tag{2.5}$$

The condition (2.4) precludes trading strategies with shortfall exceeding the value of the contract. It is intuitively clear that the initial capital and the shortfall constraint should be related to each other, that is portfolios with a restricted shortfall should keep the initial cost at a sufficiently high level. Indeed, due to the positivity of L , we have

$$(H - X_T^{x,\pi})^+ \leq L \iff H - X_T^{x,\pi} \leq L,$$

which implies that under (2.4) the condition (2.5) is equivalent to $P(X_T^{x,\pi} \geq H - L) = 1$. Hence (x, π) hedges the claim $H - L$ and consequently

$$x \geq p(H - L). \tag{2.6}$$

Below we give some natural examples of shortfall constraints corresponding to various forms of the trader's risk aversion.

Examples

- a) If $L = H$ then (2.5) boils down to the positivity of $X_T^{x,\pi}$ and hence the profile of the shortfall remains unconstrained. This case corresponds to the classical framework considered in the literature.
- b) For $L = 0$ the trader is expected to hedge the claim H , so no shortfall is acceptable at all.

- c) The trader can cover the arising portfolio loss providing that it lies below a fixed constant level $c > 0$. The maximal value of c is defined by the solvency of the trader. In this case we set

$$L = c \wedge H.$$

- d) Generalizing the previous example, the trader may want to keep the loss below c and simultaneously hedge H in some fixed price range $[a, b]$ of the underlying stock. Then L is given by

$$L = (c \wedge H) \mathbf{1}_{\{S_T < a\}} + (c \wedge H) \mathbf{1}_{\{S_T > b\}}.$$

- e) In the subjective forecast of the trader the stock price range $(0, a)$, $(b, +\infty)$ is viewed as unrealistic and hence ruled out as source of risk. The trader's aim is to keep the shortfall below c only in the interval $[a, b]$. The related form of L is

$$L = (c \wedge H) \mathbf{1}_{\{S_T \in [a, b]\}} + H \mathbf{1}_{\{S_T \notin [a, b]\}}.$$

- f) Let $\alpha \in [0, 1]$ describe a partial recovery of the claim, i.e. the claim which is to be hedged is αH . Then L is equal to

$$L = (1 - \alpha)H.$$

Our aim is to solve the classical optimization problems (1.2), (1.3), (1.4) and (1.5) which are adapted to the new framework with a constrained shortfall profile. For a given triplet (x, H, L) , which in view of the discussion above satisfies $p(H) = \sup_{Q \in \mathcal{Q}} E^Q[H] < +\infty$, (2.4), (2.5) and (2.6), we are looking for a strategy $\pi \in \Pi$ such that (x, π) is admissible and solves

$$\begin{cases} \min_{\pi \in \Pi} r(H, X_T^{x, \pi}) \\ (i) \quad P((H - X_T^{x, \pi})^+ \leq L) = 1 \\ (ii) \quad p(H - L) \leq x < p(H). \end{cases} \quad (2.7)$$

Recall that $p(H)$ stands for the price of H and is given by (2.3). Above $r(H, X_T^{x, \pi})$ describes the shortfall risk of the pair (x, π) and (2.7) will be investigated for its four concrete forms. For

$$r(H, X_T^{x, \pi}) := P(X_T^{x, \pi} < H),$$

we obtain the *quantile hedging problem* (QH). To the *generalized quantile hedging problem* (GQH) corresponds

$$r(H, X_T^{x, \pi}) := E \left[\left(1 - \frac{X_T^{x, \pi}}{H} \right) \mathbf{1}_{\{X_T^{x, \pi} < H\}} \right],$$

and to the *weighted expected shortfall problem* (WES)

$$r(H, X_T^{x, \pi}) := E[l((H - X_T^{x, \pi})^+)].$$

The loss function in WES is state dependent, i.e. $l : \Omega \times [0, +\infty) \rightarrow [0, +\infty)$ and such that $l(\omega, \cdot)$ is continuous, increasing and $l(\omega, 0) = 0$ for each $\omega \in \Omega$. We also assume that $E[l(H)] < +\infty$, so the risk measure of any admissible strategy in WES is well defined. For the shortfall risk quantified by

$$r(H, X_T^{x,\pi}) = \rho(-(H - X_T^{x,\pi})^+),$$

where ρ is a convex risk measure, (2.7) will be denoted by CRM. Recall that $\rho : L^p \rightarrow \mathbb{R}$ with $L^p = L^p(\Omega, \mathcal{F}, P)$, $p \geq 1$ is a convex risk measure if it satisfies

- a) $Z_1 \leq Z_2 \implies \rho(Z_1) \geq \rho(Z_2)$, $Z_1, Z_2 \in L^p$,
- b) $\rho(Z + a) = \rho(Z) - a$, $a \in \mathbb{R}, Z \in L^p$,
- c) $\rho(\alpha Z_1 + (1 - \alpha)Z_2) \leq \alpha\rho(Z_1) + (1 - \alpha)\rho(Z_2)$, $\alpha \in [0, 1], Z_1, Z_2 \in L^p$.

If, additionally,

- d) $\rho(\alpha Z) = \alpha\rho(Z)$, $\alpha \geq 0, Z \in L^p$.

is satisfied then ρ is called coherent.

If $L = H$ then the constraints (2.7)(i) and (2.7)(ii) amount to the admissibility of (x, π) and consequently (2.7) becomes a classical risk minimizing problem with an unconstrained shortfall profile.

3 Optimal strategies with shortfall constraint

The following result describes an optimal strategy for the problem QH. Below A^c stands for the complement of a set A .

Proposition 3.1 *Let $p(H - L) \leq x < p(H)$. If there exists a set $\tilde{A} \in \mathcal{F}$ solving the problem*

$$\begin{cases} \max_A P(A) \\ (i) \quad p(H - L\mathbf{1}_{A^c}) \leq x, \end{cases} \quad (3.1)$$

then a hedging strategy $(\tilde{x}, \tilde{\pi})$ for the claim $\tilde{H} := H - L\mathbf{1}_{\tilde{A}^c}$ with $\tilde{x} = p(\tilde{H})$ solves QH.

Proof: Let us define a success set of a strategy (x, π) by

$$A_{x,\pi} := \{X_T^{x,\pi} \geq H\}.$$

First we show that for any strategy (x, π) satisfying (2.7)(i), (ii) we have

$$P(X_T^{x,\pi} \geq H) = P(A_{x,\pi}) \leq P(\tilde{A}).$$

Since $X_T^{x,\pi} \geq H$ on $A_{x,\pi}$ and, by (2.7)(i), $X_T^{x,\pi} \geq H - L$ a.s., it follows

$$H - L\mathbf{1}_{A_{x,\pi}^c} = H\mathbf{1}_{A_{x,\pi}} + (H - L)\mathbf{1}_{A_{x,\pi}^c} \leq X_T^{x,\pi}.$$

Using the fact that $X^{x,\pi}$ is a Q -supermartingale for each $Q \in \mathcal{Q}$, we obtain

$$E^Q[H - L\mathbf{1}_{A_{x,\pi}^c}] \leq E^Q[X_T^{x,\pi}] \leq x, \quad Q \in \mathcal{Q},$$

which, by passing to supremum over $Q \in \mathcal{Q}$, gives

$$p(H - L\mathbf{1}_{A_{x,\pi}^c}) \leq x,$$

and (3.1) (i) follows. Hence $P(A_{x,\pi}) \leq P(\tilde{A})$.

Now let us consider the strategy $(\tilde{x}, \tilde{\pi})$ and notice that the condition $X_T^{\tilde{x}, \tilde{\pi}} \geq H - L\mathbf{1}_{\tilde{A}^c}$ implies

$$X_T^{\tilde{x}, \tilde{\pi}} \geq H\mathbf{1}_{\tilde{A}} + (H - L)\mathbf{1}_{\tilde{A}^c} \geq H - L, \quad (3.2)$$

It follows that $(H - X_T^{\tilde{x}, \tilde{\pi}})^+ \leq L$, which is (2.7)(i), and that $\tilde{x} \geq p(H - L)$ which together with the condition $\tilde{x} = p(\tilde{H}) \leq x$ gives (2.7)(ii). Further, it follows from (3.2) that $A_{\tilde{x}, \tilde{\pi}} \supseteq \tilde{A}$ and thus $P(A_{\tilde{x}, \tilde{\pi}}) \geq P(\tilde{A})$. Hence $A_{\tilde{x}, \tilde{\pi}} = \tilde{A}$ and the optimality of $(\tilde{x}, \tilde{\pi})$ follows. \square

To deal with the succeeding problems we will need the success ratio of an admissible strategy (x, π) defined by

$$\varphi_{x,\pi} := 1 \wedge \frac{X_T^{x,\pi}}{H}, \quad (3.3)$$

and a family of all statistical tests defined by

$$\mathcal{R} := \{\varphi : \varphi \text{ is } \mathcal{F} - \text{measurable and } 0 \leq \varphi \leq 1\}. \quad (3.4)$$

It follows from (2.4) follows that $\frac{H-L}{H} \in \mathcal{R}$ provided that $\frac{H-L}{H}$, by definition, is equal to zero on the set $\{H = 0\}$. An optimal strategy for WES is characterized by the following result.

Theorem 3.2 *Assume that the initial capital x satisfies $p(H - L) \leq x < p(H)$. Let $\tilde{\varphi} \in \mathcal{R}$ be a solution of the problem*

$$\begin{cases} \min_{\varphi} E[l((1 - \varphi)H)] \\ (i) \quad \varphi \geq \frac{H-L}{H}, \\ (ii) \quad p(H\varphi) \leq x. \end{cases} \quad (3.5)$$

Let $(\tilde{x}, \tilde{\pi})$, $\tilde{x} = p(\tilde{H})$, be a hedging strategy for the claim $\tilde{H} := H\tilde{\varphi}$. Then $(\tilde{x}, \tilde{\pi})$ solves WES and $\varphi_{\tilde{x}, \tilde{\pi}} = \tilde{\varphi}$.

Proof: Using the same type of arguments as in the proof of Proposition 3.1 in [10] we prove that $\tilde{\varphi}$ exists. Let $\{\varphi\}_n$ be a minimizing sequence satisfying (3.5)(i), (ii). There exists a new minimizing sequence

$$\tilde{\varphi}_n \in \text{conv}\{\varphi_n, \varphi_{n+1}, \dots\} \quad (3.6)$$

which converges almost surely to a limit $\tilde{\varphi}$. Since $\tilde{\varphi}_n$ is an element of the convex hull (3.6) consisting of elements satisfying (3.5)(i), (ii) and since $p(\alpha H\varphi_k + \beta H\varphi_l) \leq \alpha p(H\varphi_k) + \beta p(H\varphi_l)$, $l, k \geq$

$1, \alpha, \beta \geq 0$ holds it follows that $\tilde{\varphi}_n$ also satisfies (3.5)(i), (ii). Hence $\tilde{\varphi}$ also satisfies (3.5)(i), (ii), so it solves (3.5).

Let (x, π) be a strategy satisfying (2.7)(i), (ii). It follows from (2.7)(i) that

$$\begin{aligned}\varphi_{x,\pi} &= 1 \wedge \frac{X_T^{x,\pi}}{H} \geq \mathbf{1}_{\{X_T^{x,\pi} \geq H\}} + \frac{H-L}{H} \mathbf{1}_{\{X_T^{x,\pi} < H\}} \\ &\geq \frac{H-L}{H},\end{aligned}$$

which implies that $\varphi_{x,\pi}$ satisfies (3.5)(i). Moreover, it follows from the inequality

$$H\varphi_{x,\pi} = H \wedge X_T^{x,\pi} \leq X_T^{x,\pi},$$

and from the fact that $X^{x,\pi}$ is a Q -supermartingale for each $Q \in \mathcal{Q}$ that

$$p(H\varphi_{x,\pi}) \leq p(X_T^{x,\pi}) \leq x.$$

This means that $\varphi_{x,\pi}$ satisfies (3.5)(ii). It follows that

$$E[l((H - X_T^{x,\pi})^+)] = E[l((1 - \varphi_{x,\pi})H)] \geq E[l((1 - \tilde{\varphi})H)]. \quad (3.7)$$

Now let us focus on the strategy $(\tilde{x}, \tilde{\pi})$. Since $\tilde{\varphi}$ satisfies (3.5)(i) it follows that

$$X_T^{\tilde{x}, \tilde{\pi}} \geq H\tilde{\varphi} \geq H \cdot \frac{H-L}{H} = H-L, \quad (3.8)$$

and we obtain that $(H - X_T^{\tilde{x}, \tilde{\pi}})^+ \leq L$, which is (2.7)(i). Furthermore, (3.8) implies that $\tilde{x} \geq p(H-L)$, which together with the condition $\tilde{x} \leq x$ yields (2.7)(ii). The success ratio of $(\tilde{x}, \tilde{\pi})$ satisfies

$$\varphi_{\tilde{x}, \tilde{\pi}} = 1 \wedge \frac{X_T^{\tilde{x}, \tilde{\pi}}}{H} \geq \mathbf{1}_{\{X_T^{\tilde{x}, \tilde{\pi}} \geq H\}} + \tilde{\varphi} \mathbf{1}_{\{X_T^{\tilde{x}, \tilde{\pi}} < H\}} \geq \tilde{\varphi},$$

and from the monotonicity of l we obtain

$$E[l((H - X_T^{\tilde{x}, \tilde{\pi}})^+)] = E[l((1 - \varphi_{\tilde{x}, \tilde{\pi}})H)] \leq E[l((1 - \tilde{\varphi})H)]. \quad (3.9)$$

The result follows from (3.7) and (3.9). \square

The form of solution to GQH can be deduced from that of WES with the loss function

$$\hat{l}(\omega, z) := \frac{z}{H(\omega)} \mathbf{1}_{\{H(\omega) > 0\}}.$$

Then, for any admissible strategy (x, π) , we have $E[\hat{l}((H - X_T^{x,\pi})^+)] = E[\hat{l}((1 - \varphi_{x,\pi})H)] = E[1 - \varphi_{x,\pi}]$ and both problems GQH and WES are equivalent. This leads to the following result.

Theorem 3.3 *Let x be an arbitrary initial capital satisfying $p(H-L) \leq x < p(H)$. Denote by $\tilde{\varphi} \in \mathcal{R}$ a solution of the problem*

$$\begin{cases} \max_{\varphi} E[\varphi] \\ \varphi \geq \frac{H-L}{H}, \\ p(H\varphi) \leq x. \end{cases} \quad (3.10)$$

Then a hedging strategy $(\tilde{x}, \tilde{\pi})$ with $\tilde{x} = p(\tilde{H})$ for the payoff $\tilde{H} := H\tilde{\varphi}$ is optimal for the problem GQH and $\varphi_{\tilde{x}, \tilde{\pi}} = \tilde{\varphi}$.

The arguments in the poof of Theorem 3.2 can be successfully applied in the case when the shortfall risk is given by

$$r(H, X_T^{x,\pi}) = \rho(-(H - X_T^{x,\pi})^+),$$

where ρ stands for a convex risk measure on $L^p, p \geq 1$. Since each convex measure on L^p is pointwise continuous, see Theorem 3.1 in [11], it follows that the pointwise convergent minimizing sequence $\{\varphi_n\}$ for the problem

$$\min_{\varphi} \rho(-(1 - \varphi)H), \quad \varphi \geq \frac{H - L}{H}, \quad p(H\varphi) \leq x,$$

satisfies

$$\rho(-(1 - \varphi_n)H) \xrightarrow{n} \rho(-(1 - \tilde{\varphi})H),$$

where $\tilde{\varphi} := \lim \varphi_n$. Following the proof of Theorem 3.2 one can show that a hedging strategy for $H\tilde{\varphi}$ is optimal. Hence we obtain the following Corollary, which is a generalization of Theorem 1.5 in [14] dealing with minimizing coherent risk measures in the class of strategies with no shortfall constraints.

Corollary 3.4 *Let $1 \leq p < +\infty$. Assume that $E[H^p] < +\infty$ and ρ is a convex risk measure on L^p . For x satisfying $p(H - L) \leq x < p(H)$ let $\tilde{\varphi} \in \mathcal{R}$ be a solution of the problem*

$$\begin{cases} \min_{\varphi} \rho(-(1 - \varphi)H) \\ \varphi \geq \frac{H-L}{H}, \\ p(H\varphi) \leq x. \end{cases} \quad (3.11)$$

Let $(\tilde{x}, \tilde{\pi})$, $\tilde{x} = p(\tilde{H})$, be a hedging strategy for the claim $\tilde{H} := H\tilde{\varphi}$. Then $(\tilde{x}, \tilde{\pi})$ solves CRM.

We close this section with a remark on risk-independent hedging.

Remark 3.5 *For a given shortfall constraint L and x such that $p(H - L) \leq x < p(H)$ the investor can be interested in finding a new shortfall constraints $\tilde{L} \leq L$ and a feasible portfolio for \tilde{L} , i.e. satisfying $P((H - X_T^{x,\pi})^+ \leq \tilde{L}) = 1$. Portfolios which are not feasible for \tilde{L} are more risky than those feasible for \tilde{L} regardless of the risk measure of the trader provided that it is monotone. Since the new constraint should preserve the profile of the original one, we can search it in the class $\{L_\alpha := \alpha L, \alpha \in [0, 1]\}$. This leads to the problem*

$$\begin{cases} \min \alpha \\ P((H - X_T^{x,\pi})^+ \leq L_\alpha) = 1, \\ p(H - L_\alpha) \leq x < p(H). \end{cases} \quad (3.12)$$

Since the function

$$g(\alpha) := p(H - L_\alpha) = \sup_{Q \in \mathcal{Q}} E^Q[H - L_\alpha], \quad \alpha \in [0, 1],$$

is continuous and monotone with $g(0) = p(H)$ and $g(1) = p(H - L)$, there exist solutions of the equation $g(\alpha) = x$ and $\hat{\alpha} := \min\{\alpha \in [0, 1] : g(\alpha) = x\}$. The solution of (3.12) is $\hat{\alpha}$ and a feasible strategy for $L_{\hat{\alpha}}$ is a hedge for $H - \hat{\alpha}L$.

4 Generalized Neyman-Pearson lemma and complete markets

In this section we analyse the conditions describing the success ratios of optimal strategies considered in Section 3. Notice that using (2.3), which defines the superhedging price of H , the problem (3.10) can be written in the form

$$\begin{cases} (i) & \max_{\varphi} E^P[\varphi] \\ (ii) & \varphi^* \leq \varphi \leq 1, \\ (iii) & \sup_{\hat{Q} \in \hat{\mathcal{Q}}} E^{\hat{Q}}[\varphi] \leq x. \end{cases} \quad (4.1)$$

where $\varphi^* := \frac{H-L}{H}$ and $\hat{\mathcal{Q}}$ is the family of finite measures defined by $d\hat{Q} := HdQ$, $Q \in \mathcal{Q}$. The conditions (4.1) (i) and (4.1)(iii) correspond to the classical problem of testing a null composite hypothesis represented by the family $\hat{\mathcal{Q}}$ against a simple alternative hypothesis given by the measure P . More precisely, (4.1)(iii) is a constraint for the type I statistical error while (4.1)(i) describes minimization of the type II statistical error. The non-standard condition is (4.1)(ii) which tells that each test must exceed the minimal threshold φ^* of rejecting the null hypothesis. We call tests satisfying (4.1)(ii), (iii) *conditional tests* with a *rejection threshold* φ^* . The rejection threshold affects of course both statistical errors. The error of the first kind is bounded from below, i.e.

$$\sup_{\hat{Q} \in \hat{\mathcal{Q}}} E^{\hat{Q}}[\varphi] \geq \sup_{\hat{Q} \in \hat{\mathcal{Q}}} E^{\hat{Q}}[\varphi^*],$$

while the error of the second kind is bounded from above, i.e.

$$E^P[1 - \varphi] \leq E^P[1 - \varphi^*].$$

It follows, in particular, that (4.1) is well posed if $x \geq \sup_{\hat{Q} \in \hat{\mathcal{Q}}} E^{\hat{Q}}[\varphi^*]$. The special case when $\hat{\mathcal{Q}}$ is a singleton is of prime importance because it corresponds to complete markets which are analytically tractable. If this is the case and $\varphi^* = 0$ then (4.1) becomes a classical testing problem with simple hypotheses and its solution is described by the Neyman-Pearson lemma. There are several results in the literature which extend the classical Neyman-Pearson lemma to composite hypotheses, see [5], [16], [17], [19]. The result proven below sets up a new kind of generalization concerned with conditional tests for simple hypotheses.

Recall from (3.4), that \mathcal{R} stands for the family of statistical tests.

Lemma 4.1 *Let P and Q be any two equivalent probability measures. For given $\varphi^* \in \mathcal{R}$ and $\alpha \in [E^Q[\varphi^*], 1]$ a solution $\tilde{\varphi}$ of the problem*

$$\begin{cases} \max_{\varphi} E^P[\varphi] \\ (i) & \varphi^* \leq \varphi \leq 1, \\ (ii) & E^Q[\varphi] \leq \alpha, \end{cases} \quad (4.2)$$

has the form

$$\tilde{\varphi} = \mathbf{1}_{\{\varphi^*=1\} \cup \{\frac{dP}{dQ} > k\}} + [\varphi^* + \gamma(1 - \varphi^*)] \mathbf{1}_{\{\frac{dP}{dQ} = k\}} + \varphi^* \mathbf{1}_{\{\frac{dP}{dQ} < k\}}, \quad (4.3)$$

where $k \geq 0, \gamma \in [0, 1]$ are constants such that $E^Q[\tilde{\varphi}] = \alpha$.

Proof: It is clear that $\tilde{\varphi} = \varphi^* = 1$ on the set $\{\varphi^* = 1\}$. On the set $\{\varphi^* < 1\}$ the optimal solution $\tilde{\varphi}$ solves the problem

$$\begin{cases} \max_{\varphi} E^P[\varphi \mathbf{1}_{\{\varphi^* < 1\}}] \\ (i) \quad \varphi^* \mathbf{1}_{\{\varphi^* < 1\}} \leq \varphi \mathbf{1}_{\{\varphi^* < 1\}} \leq 1, \\ (ii) \quad E^Q[\varphi \mathbf{1}_{\{\varphi^* < 1\}}] \leq \alpha - P(\varphi^* = 1). \end{cases} \quad (4.4)$$

For any φ such that $\varphi^* \leq \varphi \leq 1$ consider the transformation

$$\Phi = \Phi(\varphi) := \frac{\varphi \mathbf{1}_{\{\varphi^* < 1\}} - \varphi^* \mathbf{1}_{\{\varphi^* < 1\}}}{(1 - \varphi^*) \mathbf{1}_{\{\varphi^* < 1\}}}, \quad (4.5)$$

which defines a random variable on the set $\hat{\Omega} := \{\varphi^* < 1\}$. The problem (4.4) can be transformed with the use of two auxiliary probability measures on $\hat{\Omega}$ with densities

$$\frac{d\hat{P}}{dP} := \frac{(1 - \varphi^*) \mathbf{1}_{\{\varphi^* < 1\}}}{E^P[(1 - \varphi^*) \mathbf{1}_{\{\varphi^* < 1\}}]}, \quad \frac{d\hat{Q}}{dQ} := \frac{(1 - \varphi^*) \mathbf{1}_{\{\varphi^* < 1\}}}{E^Q[(1 - \varphi^*) \mathbf{1}_{\{\varphi^* < 1\}}]},$$

to the form

$$\begin{cases} \max_{\Phi} E^{\hat{P}}[\Phi] \\ (i) \quad 0 \leq \Phi \leq 1, \\ (ii) \quad E^{\hat{Q}}[\Phi] \leq \frac{\alpha - Q(\varphi^* = 1) - E^Q[(1 - \varphi^*) \mathbf{1}_{\{\varphi^* < 1\}}]}{E^Q[(1 - \varphi^*) \mathbf{1}_{\{\varphi^* < 1\}}]}. \end{cases} \quad (4.6)$$

The problem (4.6) is a standard testing problem and the classical Neyman-Pearson lemma provides its solution

$$\tilde{\Phi} = \mathbf{1}_{\{\frac{d\hat{P}}{d\hat{Q}} > k\}} + \gamma \mathbf{1}_{\{\frac{d\hat{P}}{d\hat{Q}} = k\}}, \quad (4.7)$$

where $k \geq 0, \gamma \in [0, 1]$ are constants such that (4.6) (ii) holds as equality. Since

$$\frac{d\hat{P}}{d\hat{Q}} = \text{const.} \cdot \frac{dP}{dQ} \mathbf{1}_{\{\varphi^* < 1\}}, \quad \text{const.} > 0,$$

the optimal solution of (4.6) can be written in the form

$$\tilde{\Phi} = \mathbf{1}_{\{\frac{dP}{dQ} \mathbf{1}_{\{\varphi^* < 1\}} > k\}} + \gamma \mathbf{1}_{\{\frac{dP}{dQ} \mathbf{1}_{\{\varphi^* < 1\}} = k\}}, \quad (4.8)$$

where the constant k in (4.7) and (4.8) may differ. Coming back to (4.5) we determine $\tilde{\varphi} \mathbf{1}_{\{\varphi^* < 1\}}$ from the equation

$$\tilde{\Phi} = \tilde{\Phi}(\tilde{\varphi}) = \mathbf{1}_{\{\frac{dP}{dQ} \mathbf{1}_{\{\varphi^* < 1\}} > k\}} + \gamma \mathbf{1}_{\{\frac{dP}{dQ} \mathbf{1}_{\{\varphi^* < 1\}} = k\}},$$

which gives

$$\tilde{\varphi} \mathbf{1}_{\{\varphi^* < 1\}} = \mathbf{1}_{\{\frac{dP}{dQ} > k\}} + [\varphi^* + \gamma(1 - \varphi^*)] \mathbf{1}_{\{\frac{dP}{dQ} = k\}} + \varphi^* \mathbf{1}_{\{\frac{dP}{dQ} < k\}}.$$

This, in view of the decomposition $\tilde{\varphi} = \tilde{\varphi} \mathbf{1}_{\{\varphi^* = 1\}} + \tilde{\varphi} \mathbf{1}_{\{\varphi^* < 1\}}$, yields (4.3). \square

One can check that Lemma 4.1 with $\varphi^* = 0$ boils down to the classical Neyman-Pearson lemma.

The following part of this section is concerned with a precise characterization of solutions to problems (3.1), (3.5), (3.10), (3.12) in the case when the market is complete. Let us start with an auxiliary technical result.

Proposition 4.2 *Let $X \geq 0$, $Y \geq 0$ be random variables with $E[Y] < +\infty$ and such that the cumulative distribution function*

$$F_X(t) := P(X \leq t), \quad t > 0,$$

is continuous. Then the function

$$g(k) := E[Y \mathbf{1}_{\{X < k\}}], \quad k > 0,$$

is continuous.

Proof: If $Y = 0$ then the result is obvious. For $Y \neq 0$ let us consider the measure \hat{P} given by $\frac{d\hat{P}}{dP} = \frac{Y}{E[Y]}$, which is clearly absolutely continuous with respect to P . For $k > 0$ we have

$$\left| g(k + \frac{1}{n}) - g(k) \right| = E[Y \mathbf{1}_{\{k \leq X < k + \frac{1}{n}\}}] = E[Y] \hat{P} \left(k \leq X < k + \frac{1}{n} \right) \xrightarrow{n \rightarrow +\infty} E[Y] \hat{P}(X = k) = 0,$$

where the last equality follows from the absolute continuity of \hat{P} with respect to P and the continuity of F_X . The left continuity of g follows from

$$|g(k) - g(k - \frac{1}{n})| = E[Y \mathbf{1}_{\{k - \frac{1}{n} \leq X < k\}}] \xrightarrow{n \rightarrow +\infty} 0, \quad k > 0,$$

which is a consequence of the dominated convergence. \square

Now we are ready to formulate conditions which ensure the existence of solutions to problems (3.1), (3.5), (3.10), (3.12) and give their explicit form.

Proposition 4.3 *Assume that the market model is complete and denote by $Z := \frac{dQ}{dP} > 0$ the density of the unique equivalent martingale measure Q .*

a) *Assume that the function $F_{ZL}(t) = P(ZL \leq t)$, $t > 0$, is continuous. Then (3.1) has a solution \tilde{A} of the form*

$$\begin{aligned} \tilde{A} &= \{ZL < k\}, \quad \text{if } E^Q[H - L] < x < E^Q[H], \\ \tilde{A} &= \emptyset, \quad \text{if } E^Q[H - L] = x, \end{aligned} \tag{4.9}$$

where k is a constant solving the equation $E[ZL \mathbf{1}_{\{ZL < k\}}] = x - E^Q[H - L]$.

b) *Assume that the function $F_{ZH}(t) = P(ZH \leq t)$, $t > 0$, is continuous. Then the solution of (3.10) is given by*

$$\begin{aligned} \tilde{\varphi} &= \frac{H}{H - L} \quad \text{if } E^Q[H - L] = x, \\ \tilde{\varphi} &= \mathbf{1}_{\{L=0\} \cup \{ZH < k\}} + \frac{H - L}{L} \mathbf{1}_{\{ZH > k\}} \quad \text{if } E^Q[H - L] < x < E^Q[H], \end{aligned} \tag{4.10}$$

with the constant k solving

$$E[ZL \mathbf{1}_{\{ZH > k\}}] = E^Q[H] - x. \tag{4.11}$$

c) If the function $F_Z(t) = P(Z \leq t)$, $t \in \mathbb{R}$, is continuous then the solution of (3.5) with $l(z) = z$ equals

$$\begin{aligned}\tilde{\varphi} &= \frac{H}{H-L} \quad \text{if } E^Q[H-L] = x, \\ \tilde{\varphi} &= \mathbf{1}_{\{L=0\} \cup \{Z < k\}} + \frac{H-L}{L} \mathbf{1}_{\{Z > k\}} \quad \text{if } E^Q[H-L] < x < E^Q[H].\end{aligned}$$

d) The solution of (3.12) is equal to

$$L_{\tilde{\alpha}} = \frac{E^Q[H] - x}{E^Q[L]} L.$$

Proof: a) By rearranging terms and introducing the auxiliary measure \hat{Q} defined by $d\hat{Q} := \frac{L}{E^Q[L]} dQ$ one can reformulate (3.1) to the form

$$\begin{cases} \max_A P(A) \\ (i) \quad \hat{Q}[A] \leq \frac{x - E^Q[H-L]}{E^Q[L]}. \end{cases} \quad (4.12)$$

If $x = E^Q[H-L]$ then the solution of (4.12) is $\tilde{A} = \emptyset$. If $E^Q[H-L] < x < E^Q[H]$ then $0 < \frac{x - E^Q[H-L]}{E^Q[L]} < 1$ and from the classical Neyman-Pearson lemma it follows that the solution of (4.12) has the form

$$\tilde{A} = \left\{ \frac{dP}{d\hat{Q}} > c \right\} = \left\{ ZL < \frac{E^Q[L]}{c} \right\},$$

providing that the constant $c > 0$ solves the equation

$$\hat{Q}(\tilde{A}) = \frac{1}{E^Q[L]} E \left[ZL \mathbf{1}_{\{ZL < \frac{E^Q[L]}{c}\}} \right] = \frac{x - E^Q[H-L]}{E^Q[L]}. \quad (4.13)$$

Now we argue that (4.13) actually has a solution. In view of Proposition 4.2 the function

$$g(c) := \frac{1}{E^Q[L]} E \left[ZL \mathbf{1}_{\{ZL < \frac{E^Q[L]}{c}\}} \right],$$

is continuous on $(0, +\infty)$. Since $\lim_{c \rightarrow 0} g(c) = 1$ and $\lim_{c \rightarrow +\infty} g(c) = 0$ the existence of a solution to $g(c) = \frac{x - E^Q[H-L]}{E^Q[L]}$ follows. Equation (4.9) holds with $k := \frac{E^Q[L]}{c}$.

b) For $x = E^Q[H-L]$ we obtain immediately that $\tilde{\varphi} = \frac{H-L}{H}$ solves (3.3). Let us consider the case with $E^Q[H-L] < x < E^Q[H]$ and reformulate (3.3) to the form required in Lemma 4.1, that is

$$\begin{cases} \max \varphi E[\varphi] \\ (i) \quad \varphi \geq \frac{H-L}{H}, \\ (ii) \quad E^{\tilde{Q}}[\varphi] \leq \frac{x}{E^Q[H]}, \end{cases}$$

with $\frac{d\bar{Q}}{dQ} := \frac{H}{E^Q[H]}$. Since $F_{ZH}(\cdot)$ is continuous and hence $\frac{dP}{dQ} = \frac{E^Q[H]}{ZH}$ atom-free distributed, it follows that the form of solution (4.3) given by Lemma 4.1 can be reduced to

$$\begin{aligned}\tilde{\varphi} &= \mathbf{1}_{\{\frac{H-L}{H}=1\} \cup \{\frac{dP}{dQ} > c\}} + \frac{H-L}{H} \mathbf{1}_{\{\frac{dP}{dQ} < c\}} \\ &= \mathbf{1}_{\{L=0\} \cup \{ZH < \frac{E^Q[H]}{c}\}} + \frac{H-L}{H} \mathbf{1}_{\{ZH > \frac{E^Q[H]}{c}\}},\end{aligned}$$

with the constant c solving $E^{\tilde{Q}}[\tilde{\varphi}] = \frac{x}{E^{\tilde{Q}}[H]}$. The latter equation can be written in the form

$$E[ZL \mathbf{1}_{\{ZH > \frac{E^Q[H]}{c}\}}] = E^Q[H] - x,$$

which is (4.11). Notice that (4.11) has a solution because $0 < E^Q[H] - x < E^Q[L]$ holds, the function

$$h(c) := E[ZL \mathbf{1}_{\{ZH > \frac{E^Q[H]}{c}\}}], \quad c > 0,$$

is continuous by Proposition 4.2 and satisfies

$$\lim_{c \rightarrow 0} h(c) = 0, \quad \lim_{c \rightarrow +\infty} h(c) = E[ZL \mathbf{1}_{\{H > 0\}}] = E[ZL] - E[ZL \mathbf{1}_{\{H=0\}}] = E[ZL] = E^Q[L].$$

Setting $k := \frac{E^Q[H]}{c}$ we obtain the assertion.

c) With two auxiliary measures \bar{P}, \bar{Q} given by $\frac{d\bar{P}}{dP} = \frac{H}{E[H]}$ and $\frac{d\bar{Q}}{dQ} = \frac{H}{E^Q[H]}$ one can mimic the proof of (b).

d) Follows immediately because $\tilde{\alpha}$ solves the equation $E^Q[H - \tilde{\alpha}L] = x$. \square

5 Concrete complete markets

Our aim now is to minimize the hedging risk of a call option $(S_T - K)^+, K > 0$ in the class of strategies subject to the shortfall constraint $L = c \wedge (S_T - K)^+$ with $c \geq 0$. The Black-Scholes and exponential Poisson models will be examined. The initial capital of the investor is assumed to satisfy (2.6), which amounts to

$$p\left((S_T - (K + c))^+\right) \leq x < p\left((S_T - K)^+\right).$$

This means that x is less than the replicating cost of the option but is also greater than the replicating cost of the call option with strike $K + c$. Combining Proposition 3.1, Theorem 3.3, Theorem 3.2 together with Lemma 4.1 and Proposition 4.3 we show in the following paragraphs that an optimal strategy hedges always an option which is a sum of two knock-out options, i.e. it has the form

$$\tilde{H} = (S_T - K)^+ \mathbf{1}_A + (S_T - (K + c))^+ \mathbf{1}_{A^c}, \quad (5.1)$$

where $A \in \mathcal{F}_T$ is a set which depends on the initial capital x and the risk measure of the investor. For the exponential Poisson model an additional term in (5.1) appears which is related to the presence of jumps of the price process, see formula (5.10) below.

5.1 Black-Scholes model

Let us recall some basics concerning the Black-Scholes model. The asset price dynamics has the form

$$dS_t = S_t(\alpha dt + \sigma dW_t), \quad S_0 = s_0, \quad t \in [0, T], \quad \alpha \in \mathbb{R}, \sigma > 0.$$

The unique martingale measure Q is given by

$$\frac{dQ}{dP} = Z = e^{-\theta W_T - \frac{1}{2}\theta^2 T},$$

with $\theta = \frac{\alpha}{\sigma}$. Under Q the process $\tilde{W}_t := W_t + \theta t$ is a Wiener process and the dynamics of S under Q has the form $dS_t = \sigma d\tilde{W}_t$. The price of the call option is given by

$$C_{BS}(K) := p((S_T - K)^+) = E^Q[(S_T - K)^+] = s_0\phi(d_1) - K\phi(d_2),$$

where

$$d_1 := \frac{\ln\left(\frac{s_0}{K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 := d_1 - \sigma\sqrt{T},$$

and Φ stands for the $N(0, 1)$ -cumulative distribution function.

Below we solve the problems QH, GQH and WES explicitly. First notice that if $x = C_{BS}(K + c)$ then QH, GQH and WES have the same solution which is the replicating strategy for the payoff $(S_T - (K + c))^+$. Hence in the sequel we consider the case $C_{BS}(K + c) < x < C_{BS}(K)$. For the sake of simplicity the parameters are assumed to satisfy $0 < \alpha < \sigma^2$. Another cases can be treated, however, in a similar way.

Quantile hedging problem (QH)

Proposition 5.1 *Let $C_{BS}(K + c) < x < C_{BS}(K)$. An optimal strategy for a call option $(S_T - K)^+$ with the shortfall constraint $L = c \wedge (S_T - K)^+$ in the Black-Scholes model with parameters satisfying $0 < \alpha < \sigma^2$ is a replicating strategy for the payoff*

$$\tilde{H} = (S_T - K)^+ \mathbf{1}_{\{S_T \leq I(k)\}} + (S_T - K)^+ \mathbf{1}_{\{S_T \geq J(k)\}} + (S_T - (K + c))^+ \mathbf{1}_{\{I(k) < S_T < J(k)\}}, \quad (5.2)$$

where

$$I(k) := \hat{y}(k) \wedge (K + c),$$

$$J(k) := (Ck)^{\frac{\sigma^2}{\alpha}} \vee (K + c),$$

with $C := \left(\frac{1}{s_0} e^{-\frac{1}{2}(\alpha + \sigma^2)T}\right)^{-\frac{\alpha}{\sigma^2}}$ and $\hat{y}(k)$ being the unique solution of the equation

$$\frac{1}{Ck} y^{\frac{\alpha}{\sigma^2}} = y - K, \quad y \geq 0,$$

The constant k in (5.2) is uniquely defined by the relation

$$C_{BS}(K) + C_{BS}(K + c) - C_{BS}(I(k)) - (I(k) - K)(1 - \Phi(e_1(k))) + c(1 - \Phi(e_2(k))) = x, \quad (5.3)$$

where

$$e_1(k) := \frac{\ln\left(\frac{I(k)}{s_0}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad e_2(k) := \frac{\ln\left(\frac{J(k)}{s_0}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

In particular, the shortfall of the optimal strategy equals $[S_T \wedge (K + c) - K]\mathbf{1}_{\{I(k) < S_T < J(k)\}}$.

Proof: Since $Z = CS_T^{-\frac{\alpha}{\sigma^2}}$ and $0 < \alpha < \sigma^2$ it follows from Proposition 4.3 (a) that the form of a solution to (3.1) is

$$\begin{aligned} \tilde{A} &= \{ZL < \frac{1}{k}\} = \{S_T \geq (kcC)^{\frac{\sigma^2}{\alpha}} \vee (K + c)\} \cup \{S_T \leq \hat{y}(k) \wedge (K + c)\} \\ &= \{S_T \leq I(k)\} \cup \{S_T \geq J(k)\}. \end{aligned}$$

Hence the optimal payoff given by Proposition 3.1 is

$$\begin{aligned} \tilde{H} &= \tilde{H}(k) = (S_T - K)^+ \mathbf{1}_{\tilde{A}} + \left((S_T - K)^+ - (S_T - K)^+ \wedge c\right) \mathbf{1}_{\tilde{A}^c} \\ &= (S_T - K)^+ \mathbf{1}_{\{S_T \leq I(k)\}} + (S_T - K)^+ \mathbf{1}_{\{S_T \geq J(k)\}} + (S_T - (K + c))^+ \mathbf{1}_{\{I(k) < S_T < J(k)\}}. \end{aligned} \quad (5.4)$$

To get an explicit characterization of k let us decompose \tilde{H} into the form

$$\tilde{H} = (S_T - K)^+ + (S_T - (K + c))^+ - (S_T - I(k))^+ - (I(k) - K)\mathbf{1}_{\{S_T > I(k)\}} + c\mathbf{1}_{\{S_T > J(k)\}},$$

which allows to determine the price of \tilde{H} in terms of the Black-Scholes call prices. Hence the equation for k is

$$C_{BS}(K) + C_{BS}(K + c) - C_{BS}(I(k)) - (I(k) - K)Q(S_T > I(k)) + cQ(S_T \geq J(k)) = x$$

which leads directly to (5.3). \square

Generalized quantile hedging problem (GQH)

Proposition 5.2 *Let $C_{BS}(K + c) < x < C_{BS}(K)$. An optimal strategy for a call option $(S_T - K)^+$ with the shortfall constraint $L = c \wedge (S_T - K)^+$ in the Black-Scholes model with parameters satisfying $0 < \alpha < \sigma^2$ is a replicating strategy for the payoff*

$$\tilde{H} = (S_T - K)^+ \mathbf{1}_{\{S_T \leq \hat{y}(k)\}} + (S_T - (K + c))^+ \mathbf{1}_{\{S_T > \hat{y}(k)\}},$$

where $\hat{y}(k)$ is defined in Proposition 5.1 and k is a solution of the equation

$$\begin{aligned} &C_{BS}(K) - C_{BS}(\hat{y}(k)) - (\hat{y}(k) - K)(1 - \Phi(e(k))) + C_{BS}(K + c) \\ &- \left[C_{BS}(K + c) - C_{BS}(\hat{y}(k)) - (\hat{y}(k) - (K + c))(1 - \Phi(e(k))) \right] \mathbf{1}_{\{K + c \leq \hat{y}(k)\}} = x \end{aligned} \quad (5.5)$$

where

$$e(k) := \frac{\ln\left(\frac{\hat{y}(k)}{s_0}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

In particular, the shortfall of the optimal strategy equals $\{S_T \wedge (K + c) - K\}\mathbf{1}_{\{S_T > \hat{y}(k)\}}$.

Proof: In view of Proposition 4.3 (b), it follows that the solution $\tilde{\varphi}$ of (3.10) has the form

$$\tilde{\varphi} = \mathbf{1}_{\{ZH < \frac{1}{k}\}} + \frac{H-L}{H} \mathbf{1}_{\{ZH > \frac{1}{k}\}}.$$

From Theorem 3.3 we obtain the optimal payoff

$$\tilde{H} = \tilde{H}(k) = H\tilde{\varphi} = H - L\mathbf{1}_{\{\frac{1}{k} < ZH\}} = (S_T - K)^+ \mathbf{1}_{\{S_T \leq \hat{y}(k)\}} + (S_T - (K + c))^+ \mathbf{1}_{\{S_T > \hat{y}(k)\}},$$

which can be also written in the form

$$\begin{aligned} \tilde{H} &= (S_T - K)^+ - (S_T - \hat{y}(k))^+ - (\hat{y}(k) - K) \mathbf{1}_{\{S_T > \hat{y}(k)\}} + (S_T - (K + c))^+ \\ &\quad - \left[(S_T - (K + c))^+ - (S_T - \hat{y}(k))^+ - (\hat{y}(k) - (K + c)) \mathbf{1}_{\{S_T > \hat{y}(k)\}} \right] \mathbf{1}_{\{K+c \leq \hat{y}(k)\}}. \end{aligned}$$

Finally the constant k is determined by

$$\begin{aligned} E^Q[\tilde{H}] &= C_{BS}(K) - C_{BS}(\hat{y}(k)) - (\hat{y}(k) - K)(1 - \Phi(e(k))) + C_{BS}(K + c) \\ &\quad - \left[C_{BS}(K + c) - C_{BS}(\hat{y}(k)) - (\hat{y}(k) - (K + c))(1 - \Phi(e(k))) \right] \mathbf{1}_{\{K+c \leq \hat{y}(k)\}}, \end{aligned}$$

which leads to (5.5). \square

Remark 5.3 *It follows from the proofs of Propositions 5.1 and 5.2 that the optimal payoffs for QH and GQH are of the forms*

$$\begin{aligned} &(S_T - K)^+ \mathbf{1}_{\{\frac{1}{k_1} > ZL\}} + (S_T - (K + c))^+ \mathbf{1}_{\{\frac{1}{k_1} < ZL\}}, \quad k_1 \geq 0, \\ &(S_T - K)^+ \mathbf{1}_{\{\frac{1}{k_2} > ZH\}} + (S_T - (K + c))^+ \mathbf{1}_{\{\frac{1}{k_2} < ZH\}}, \quad k_2 \geq 0, \end{aligned}$$

respectively, even for a general form of the shortfall constraint L . It follows that, in general, the solutions of QH and GQH differ except for the case $L = H$ which corresponds to the case with no shortfall constraint studied in [9].

Weighted expected shortfall problem (WES)

Proposition 5.4 *Let $C_{BS}(K + c) < x < C_{BS}(K)$ and $l(z) = z$. An optimal strategy for a call option $(S_T - K)^+$ with the shortfall constraint $L = c \wedge (S_T - K)^+$ in the Black-Scholes model with parameters satisfying $\alpha > 0$, $\sigma^2 > 0$ is a replicating strategy for the payoff*

$$\tilde{H} = (S_T - K)^+ \mathbf{1}_{\{S_T > k\}} + (S_T - (K + c)) \mathbf{1}_{\{S_T \leq k\}}$$

with the constant $k \geq 0$ solving

$$\begin{aligned} &C_{BS}(K) - [C_{BS}(K) - C_{BS}(k) - (k - K)(1 - \Phi(f(k)))] \mathbf{1}_{\{k > K\}} \\ &\quad + [C_{BS}(K + c) - C_{BS}(k) - (k - (K + c))(1 - \Phi(f(k)))] \mathbf{1}_{\{c > K+l\}} = x, \end{aligned} \quad (5.6)$$

where

$$f(k) := \frac{\ln(\frac{k}{s_0}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

In particular, the shortfall of the optimal strategy equals $\{S_T \wedge (K + c) - S_T \wedge K\} \mathbf{1}_{\{S_T \leq k\}}$.

Proof: By Proposition 4.3 (c) the solution of (3.5) is

$$\tilde{\varphi} = \mathbf{1}_{\{S_T > k\}} + \frac{H - L}{H} \mathbf{1}_{\{S_T \leq k\}},$$

where k is such that the corresponding optimal payoff

$$\tilde{H} = \tilde{H}(k) = H\tilde{\varphi} = H - L\mathbf{1}_{\{S_T \leq k\}} = (S_T - K)^+ \mathbf{1}_{\{S_T > k\}} + (S_T - (K + c))^+ \mathbf{1}_{\{S_T \leq k\}},$$

satisfies $E^Q[\tilde{H}] = x$. The existence and uniqueness of the constant k can be argued as before. We obtain its direct characterization by decomposing the optimal payoff \tilde{H} to the combination of the call options with maturities $k, K, K + c$ and applying the Black-Scholes formula. This leads to (5.6). \square

5.2 Exponential Poisson model

Let the asset price be given by

$$S_t = e^{N_t - \gamma t}, \quad t \in [0, T],$$

where N is a Poisson process with intensity $\lambda > 0$ under the measure P and $\gamma > 0$. The paths of S are neither increasing nor decreasing almost surely, so the model is arbitrage-free, see Theorem 3.2 in [18] or Proposition 9.9 in [3]. It is known that any equivalent measure Q is characterized by a new intensity $\lambda_Q(t) = \lambda_Q(\omega, t) \geq 0$ of N which means that the process

$$\tilde{N}_t^Q := N_t - \int_0^t \lambda_Q(s) ds, \quad t \in [0, T]$$

is a Q -martingale. Using the jump measure language it means that the jump measure of N defined by

$$\pi(t, A) := \#\{s \in [0, t] : \Delta N_s \in A\}, \quad A \subseteq \mathbb{R}, \quad 0 \notin \bar{A},$$

has a compensating measure under Q of the form

$$\nu_Q(dt, dy) := \lambda_Q(t) \mathbf{1}_{\{y=1\}}(y) dt dy,$$

that is $\tilde{\pi}_Q(dt, dy) := \pi(dt, dy) - \nu_Q(dt, dy)$ is a compensated measure under Q . The corresponding density of Q with respect to P , given by the Girsanov theorem, equals

$$\frac{dQ}{dP} = e^{\int_0^T \ln\left(\frac{\lambda_Q(s)}{\lambda}\right) dN_s - \int_0^T (\lambda_Q(s) - \lambda) ds}. \quad (5.7)$$

Let us determine the process λ_Q so that Q is a martingale measure. The Itô formula provides

$$\begin{aligned} S_t &= 1 + \int_0^t S_{s-} dN_s - \gamma \int_0^t S_{s-} ds + \sum_{s \in [0, t]} S_{s-} (e^{\Delta N_s} - 1 - \Delta N_s) \\ &= 1 - \gamma \int_0^t S_{s-} ds + \sum_{s \in [0, t]} S_{s-} (e^{\Delta N_s} - 1) \\ &= 1 - \gamma \int_0^t S_{s-} ds + \int_0^t \int_{\mathbb{R}} S_{s-} (e^y - 1) \tilde{\pi}_Q(ds, dy) + \int_0^t \int_{\mathbb{R}} S_{s-} (e^y - 1) \nu_Q(ds, dy) \\ &= 1 + \int_0^t \int_{\mathbb{R}} S_{s-} (e^y - 1) \tilde{\pi}_Q(ds, dy) + \int_0^t S_{s-} ((e - 1)\lambda_Q(s) - \gamma) ds. \end{aligned}$$

It follows that S is a local martingale under Q if and only if

$$\lambda_Q(t) \equiv \lambda_Q = \frac{\gamma}{e-1}. \quad (5.8)$$

Hence from (5.7) and (5.8) it follows that the model admits only one martingale measure Q with the density of the form

$$\frac{dQ}{dP} = Z = e^{\ln\left(\frac{\lambda_Q}{\lambda}\right)N_T - (\lambda_Q - \lambda)T} = C \cdot S_T^{\ln\left(\frac{\gamma}{\lambda(e-1)}\right)},$$

where

$$C := e^{T\left(\lambda - \frac{\gamma}{e-1} - \gamma \ln\left(\frac{\lambda(e-1)}{\gamma}\right)\right)}.$$

It follows that the price of a call option $(S_T - K)^+, K \geq 0$ is equal to

$$C_{EP}(K) := E^Q[(e^{N_T - \gamma T} - K)^+] = \sum_{n=\lceil \ln K + \gamma T \rceil}^{+\infty} (e^{-\gamma T} - K) e^{-\frac{\gamma}{e-1}T} \frac{\left(\frac{\gamma}{e-1}T\right)^n}{n!},$$

where $\lceil a \rceil := \inf\{n \in \mathbb{N} : n \geq a\}$.

In this model the quantile hedging problem does not have a solution because the set \tilde{A} described by Proposition 3.1 may not exist. Since the law of ZL is not atom-free, Proposition 4.3 can not be applied. Below we solve the generalized quantile hedging problem by direct application of Lemma 4.1 in the case when H is a call option and the coefficients satisfy $1 < \frac{\lambda(e-1)}{e} < \gamma$. Other cases can be treated in a similar manner. Notice that for $x = C_{EP}(K + c)$ the replicating strategy for $(S_T - (K + c))^+$ is optimal.

Proposition 5.5 *Let $H = (S_T - K)^+$ and $L = c \wedge (S_T - K)^+$ with $c, K \geq 0$. If*

$$1 < \frac{\lambda(e-1)}{e} < \gamma, \quad (5.9)$$

then an optimal solution to the generalized quantile hedging problem with initial capital x satisfying

$$C_{EP}(K + c) < x < C_{EP}(K)$$

is a replicating strategy for the payoff

$$\tilde{H} = (S_T - K)^+ \mathbf{1}_{\{S_T < \hat{y}(k)\}} + (S_T - (K + c))^+ \mathbf{1}_{\{S_T \geq \hat{y}(k)\}} + \gamma \left(c \wedge (S_T - K)^+ \right) \mathbf{1}_{\{S_T = \hat{y}(k)\}}. \quad (5.10)$$

Here, $\hat{y}(k)$ stands for the unique solution of the equation

$$y - K = \frac{1}{C_k} y^{\ln\left(\frac{\lambda(e-1)}{\gamma}\right)}; \quad y \geq 0,$$

the constant k is determined as

$$k := \inf \left\{ u \geq 0 : f(u) := C_{EP}(K) - C_{EP}(\hat{y}(u)) - (\hat{y}(u) - K)Q(S_T > \hat{y}(u)) + C_{EP}(K + c) \right. \\ \left. - [C_{EP}(K + c) - C_{EP}(\hat{y}(u)) - (\hat{y}(u) - (K + c))Q(S_T \geq \hat{y}(u))] \mathbf{1}_{\{K + c \leq \hat{y}(u)\}} \leq x \right\}, \quad (5.11)$$

and γ solves the equation

$$f(k) + \gamma \left(c \wedge (\hat{y}(k) - K) \right) Q(S_T = \hat{y}(k)) = x. \quad (5.12)$$

The corresponding shortfall equals

$$\left(c \wedge (S_T - K)^+ \right) \mathbf{1}_{\{S_T > \hat{y}(k)\}} + (1 - \gamma) \left(c \wedge (\hat{y}(k) - K)^+ \right) \mathbf{1}_{\{S_T = \hat{y}(k)\}}.$$

Proof: Solving (3.10) with the use of Lemma 4.1 yields the optimal payoff

$$\begin{aligned} \tilde{H} &= H\tilde{\varphi} = H\mathbf{1}_{\{\frac{dP}{dQ} > a\}} + (H - L + \gamma L)\mathbf{1}_{\{\frac{dP}{dQ} = a\}} + (H - L)\mathbf{1}_{\{\frac{dP}{dQ} < a\}} \\ &= H - L\mathbf{1}_{\{\frac{dP}{dQ} \leq a\}} + \gamma L\mathbf{1}_{\{\frac{dP}{dQ} = a\}} \\ &= (S_T - K)^+ - \left(c \wedge (S_T - K)^+ \right) \mathbf{1}_{\{\frac{1}{k} \leq Z(S_T - K)^+\}} + \gamma \left(c \wedge (S_T - K)^+ \right) \mathbf{1}_{\{\frac{1}{k} = Z(S_T - K)^+\}} \end{aligned}$$

where $\frac{d\tilde{Q}}{dQ} = \frac{H}{E^Q[H]}$ and the constants $a, k \geq 0, \gamma \in [0, 1]$ should be such that $E^Q[\tilde{H}] = x$. Using (5.9) we obtain the alternative characterization

$$\tilde{H} = \tilde{H}(k, \gamma) = (S_T - K)^+ - \left(c \wedge (S_T - K)^+ \right) \mathbf{1}_{\{S_T \geq \hat{y}(k)\}} + \gamma \left(c \wedge (S_T - K)^+ \right) \mathbf{1}_{\{S_T = \hat{y}(k)\}}.$$

To characterize k and γ let us notice that the function

$$f(z) := E^Q \left[(S_T - K)^+ - \left(c \wedge (S_T - K)^+ \right) \mathbf{1}_{\{S_T \geq \hat{y}(z)\}} \right]; \quad z \geq 0,$$

is decreasing, càdlàg and satisfies

$$\lim_{z \rightarrow 0} f(z) = C_{EP}(K), \quad \lim_{z \rightarrow +\infty} f(z) = C_{EP}(K + c), \quad (5.13)$$

$$\Delta f(z) = - \left(c \wedge (\hat{y}(z) - (K + c))^+ \right) Q(S_T = \hat{y}(z)). \quad (5.14)$$

Since $C_{EP}(K) \leq x < C_{EP}(K + c)$ and (5.13) holds, the constant $k := \inf\{z \geq 0 : f(z) \leq x\}$ is well defined. Moreover, there exists $\gamma \in [0, 1]$ such that $x - f(k) = \gamma(-\Delta f(k))$. In view of (5.14) this yields

$$x = f(k) + \gamma \left(c \wedge (\hat{y}(k) - (K + c))^+ \right) Q(S_T = \hat{y}(k)),$$

which means that $E^Q[\tilde{H}(k, \gamma)] = x$ as required. To obtain (5.11) and (5.12) one decomposes \tilde{H} into the form

$$\begin{aligned} \tilde{H} &= (S_T - K)^+ - (S_T - \hat{y}(k))^+ - (\hat{y}(k) - K) \mathbf{1}_{\{S_T > \hat{y}(k)\}} + (S_T - (K + c))^+ \\ &\quad - \left[(S_T - (K + c))^+ - (S_T - \hat{y}(k))^+ - (\hat{y}(k) - (K + c)) \mathbf{1}_{\{S_T \geq \hat{y}(k)\}} \right] \mathbf{1}_{\{K + c \leq \hat{y}(k)\}} \\ &\quad + \gamma \left(c \wedge (S_T - K)^+ \right) \mathbf{1}_{\{S_T = \hat{y}(k)\}}. \end{aligned}$$

and notices that $E^Q[\tilde{H}] = f(k) + \gamma \left(c \wedge (\hat{y}(k) - K) \right) Q(S_T = \hat{y}(k))$. □

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